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Using Laser-Induced Fluorescence in the Study of Surface Wave Turbulence

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The turbulent motion of capillary surface waves is studied using laser-induced fluorescence. A blue laser is focused onto the surface of a solution of fluorescein in water contained in a vertically shaken vessel. The movement of the resulting green spot is followed by a position-sensitive detector. The scaling behavior and cross-over phenomenon for the surface-height-frequency spectrum observed provide direct support to the weak turbulence theory of surface waves.

KEY WORDS: Surface waves; weak turbulence; laser-induced fluorescence; amplitude spectra.

INTRODUCTION

When a vessel containing fluid with a free surface is oscillated vertically, parametric excitation of surface waves may result [1]. Strongly driven, the wave motion can become chaotic. This motion is believed to be describable in terms of the weak wave turbulence (WWT) theory, which is distinguished from strong turbulence by the presence of small-amplitude waves. This presence makes a theoretical treatment feasible in terms of a perturbation theory [2], making it possible to derive the Kolmogorov spectrum characterizing the energy cascade in case of a constant energy flux P in k-space. For capillary ripples the Kolmogorov spectrum has been known for more than 30 years. However, direct measurements of the spectrum have just recently been performed.

Using a diffusing light technique, two groups, Wright *et al.* [3,4] and Henry *et al.* [5,6] have now presented measurements of the surface-height spectrum $\langle|a_w|^2\rangle$. However, the results disagree in regard to the consistency with the WWT theory. In particular, the validity of the random phase approximation underlying the WWT theory has been questioned by Wright *et al.* [3].

Apart from the nature of the diffusing agent used (suspending polystyrene spheres in water, Wright *et al.* [3,4]; semiskimmed milk, Henry *et al.* [5,6]), both previous measurements rely on the intensity of light diffusing through the fluid from below, producing a reliable image of the surface height, unlike an image obtained by the ordinary shadow-graphs technique (see e.g., [7]). From the measurements of Wright *et al.*, the instantaneous height of the entire surface is obtained, and the angularly averaged surface-height spectrum $\sim k\langle|a_\kappa|^2\rangle$ is determined. A decay exponent of -4.2 was found, which is more than 10% different from the value $-15/4$ predicted by theory. Also the frequency dependence of a 200-μ spot was measured by a light-sensitive diode, and the decay exponent for $\langle|a_w|^2\rangle$ was reported to be -3.2, consistent with the result for $k\langle|a_\kappa|^2\rangle$,[3] but again with a significant discrepancy of 13% from the theoretical value $-17/6$.

The measurements by Henry *et al.* [5,6]), on the other hand, found nearly perfect agreement with the WWT theory. Also found was a cross-over to a steeper scaling region that was related to a folding with a spatial

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[3] $\frac{k}{\omega}k^{-4.2} \rightarrow \omega^{-3.2}$ using the dispersion relation.

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average over the area in view from the detector. The slope here also agreed with the theoretical prediction.

The light-diffusion technique relies on the light mean free path being smaller than the total depth of the fluid but larger than the height of the wave structure. The diffusing agent is added to the water until these criteria are seen to be met for a laser beam passing through the water. If these criteria are not met, the connection between height and signal may be ambiguous (for a more detailed description of the method, see Wright *et al.* [3] and Ishimaru [8]).

The work by Wright *et al.* raised substantial questions regarding the WWT theory. In particular, Wright *et al.* noticed the presence of intermittency effects, leaving open the question of whether the differences between their experimental and the theoretical value for the exponents were real or just a result of experimental limitations and error bars. This paper aims at resolving this question through detailed measurements of the surface-height spectrum $\langle |a_w|^2 \rangle$. Using a fundamentally different method involving laser-induced fluorescence, there is now a clarification on these questions.

THEORETICAL BACKGROUND

In the case of the formation of waves on the surface of an ideal incompressible and deep fluid, a Hamiltonian perturbation theory can be formulated and the Kolmogorov spectrum characterizing the energy cascade in case of a constant energy flux P in k-space can be derived.

According to the weak turbulence theory of capillary ripples, the lowest-order term of the Hamiltonian has the form [2]:

$$H = \int \omega_k n_k d\mathrm{k} \qquad (1)$$

where the “number of waves” n_k in the isotropic approach is assumed only to depend on $k = |\mathbf{k}|$, i.e., $n_k = n_k$. The theory of weak turbulence has been developed quite extensively, and kinetic equations for n_k have been derived. The number of waves is related to the surface-height spectrum $\langle |a_\kappa|^2 \rangle$,

$$\langle |a_\kappa|^2 \rangle = (k/\rho\omega_k) n_k. \qquad (2)$$

For capillary ripples with dispersion relation

$$\omega = \omega_k = (\sigma/\rho)^{1/2}\, k^{3/2}, \qquad (3)$$

(σ is surface tension, ρ is density), the number of waves are given by

$$n_k \sim P^{1/2}\, \rho^{3/4}\, \sigma^{-1/4}\, k^{-17/4}, \qquad (4)$$

leading to a surface-height spectrum [9]:

$$\langle |a_\kappa|^2 \rangle \sim P^{1/2}\, \rho^{1/4}\, \sigma^{-3/4}\, k^{-19/4}. \qquad (5)$$

From this follows for the angular averaged spectrum:

$$k\langle |a_\kappa|^2 \rangle \sim P^{1/2}\, \rho^{1/4}\, \sigma^{-3/4}\, k^{-15/4}. \qquad (6)$$

In frequency representation, the spectrum is ($d\mathbf{k} \rightarrow (k^2/\omega)d\omega$)

$$\langle |a_\omega|^2 \rangle \sim P^{1/2}\, \rho^{-2/3}\, \sigma^{1/6}\, \omega^{-17/6}. \qquad (7)$$

EXPERIMENT

A circular cell with an inner diameter of 24 cm was mounted on an LDS vibration exciter system type V406/8-PA500L and driven vertically by a sinusoidal signal coming from a frequency synthesizer SRI model DS345. The cell was filled to a height of 2 cm (volume of 900 cm^3) with water mixed with sodium fluorescein. A laser beam of wavelength 488 nm from an argon-ion laser (Omnichrome 532-AP-A01, 30 mW continuous output) was directed perpendicular to the surface. A lens could be inserted in the beam to narrow the beam diameter at incidence. The amount of fluorescein is sufficient to ensure that the resulting fluorescence occurs from a volume concentrated at the surface of the liquid (penetration depth much smaller than the fluid wavelength of interest). Measurements of the bulk viscosity, density, and surface tension show only small changes from the corresponding values of clean water. Thus the qualitative behavior of the liquid is expected to be unchanged. However, the mixture has to be stirred carefully before data acquisition. Left to itself the liquid developes a concentrated layer of fluorescein at the surface, giving rise to standing wave patterns that may even show a low-frequency spiral-like oscillation.[4]

The vertically moving fluorescing green spot was viewed with a position sensitive detection system. This consisted of a microscope viewing the spot from askance and imaging the spot onto a quadrant photosensor (Hamamatsu S5981). The image was carefully centered on the detector, with no drive present. Furthermore, an interference filter (535 nm ± 5 nm) was inserted in the detection light path to block stray reflections of the incident laser

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[4] Above threshold for surface waves a regime is found, much broader in amplitude than is usually the case, where a stable square pattern exists. Inside this is a window where the wave vector in the center region starts oscillating around its average direction with a frequency of about one hertz. At the same time the pattern at the rim stays fixed. Thus the pattern oscillates in a spiral like fashion with a frequency that can be tuned by the drive amplitude.

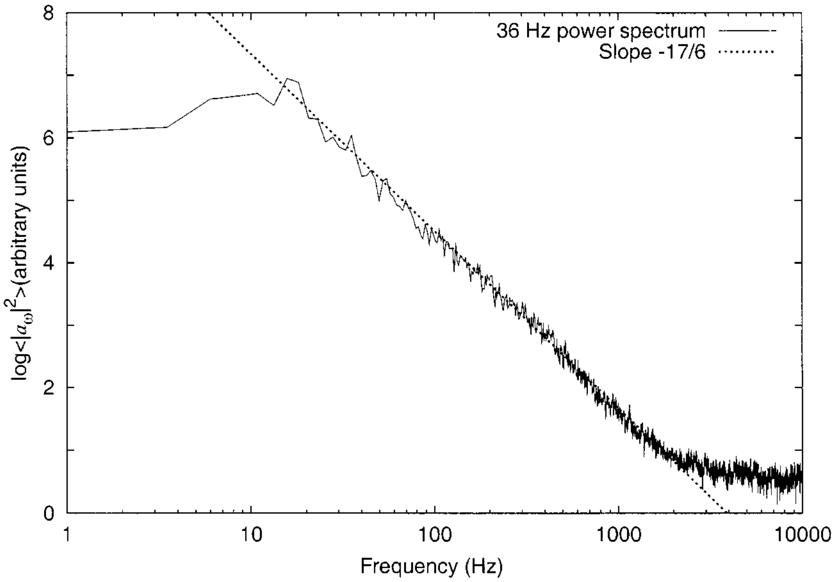

Fig. 1. Spectrum obtained at a drive frequency of 36 Hz giving a fundamental wave frequency of 18 Hz. The straight line has the slope −17/6, predicted by theory.

beam from interfering with the detection of the fluorescent signal.

The signal from the photosensor was amplified and sampled by a computer recording simultaneous time series of $2^{18} = 262144$ datapoint per channel. The dynamic range of the National Instrument AT-MIO-64E3 data acquisition board used is 12 bit. The data series obtained was carefully scanned to guard against possible overflow and spikes originating from stray reflections of the laser beam. The difference signal was corrected for changes in the total signal and for possible dc off-set. However, these corrections resulted in only minor changes in the final spectrum, mainly at very low frequencies.

RESULTS

Low-Frequency Results

Figure 1 shows the power spectra obtained at a driving frequency of $f = 36$ Hz[5] and at large driving amplitudes.[6] The diameter of the laser beam is less than 100 μm. Also shown is a straight line with slope −17/6, as predicted by WWT theory. The experimental data are found to follow this power law within a variation of less than 5%, in contrast to the 13% discrepancy obtained by Wright *et al.* but in agreement with the measurements of Henry *et al.* Bumps (as seen in the figure) and smaller deviations from the expected slope are often encountered in these spectra. This could be due to intermittency; however, when averaged over many traces the bumps tend to disappear and the slope comes out as predicted by theory.

At low frequencies the complete dispersion relation and full spectrum of wave interactions should of course be used in calculating the theoretical frequency response. Unfortunately, the theory has not yet been developed to this stage. For the frequency regime covered in this experiment, however, the error induced by only considering the capillary part is negligible.

In increasing the size of the laser beam width, a cross-over appears. The cross-over frequency can be related to a length scale through the dispersion relation. At length scales larger than the cross-over length, the beam spot on the surface can be considered point-like and the signal is then a true representation of the vertical motion of the spot. For high frequencies, (i.e., for length scales smaller than the cross-over length at which $k^2A \sim 1$), one must take into account that the signal is an average over a wave-packet [2] correlated over a distance $\sim k^{-1}$. Thus the signal recorded is $\sim k^{-2}a_\kappa$. Consequently, the scaling behavior of the spectrum given by Eq. (5) is for high frequencies replaced by the behavior

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[5] The parametric drive produces a wave frequency which is $f/2$.

[6] The driving amplitude was chosen as high as possible, requiring a non-singular surface (no drops repelled from the surface). At 36 Hz the amplitude was about three times the amplitude at which surface waves are first formed.

$$k^{-4}\,\langle|a_k|^2\rangle \sim P^{1/2}\,\rho^{1/4}\,\sigma^{-3/4}\,k^{-35/4}. \tag{8}$$

In the frequency representation ($dk \rightarrow (k^2/\omega)d\omega$), the $k^{-35/4}$ behavior corresponds to a $\omega^{-11/2}$ behavior ($k^{-35/4}k^2/\omega$ with $k \rightarrow \omega^{2/3}$). This is exactly what is observed experimentally (Fig. 2) using a beam width of 1 mm (corresponding to a fundamental wave frequency of ~ 700 Hz).

Thus the results using laser-induced fluorescence are in agreement with the results obtained by Henry *et al.* using light diffusion and confirms the weak turbulence theory.

High-Frequency Results

Earlier experiments performed at the much higher drive frequency of 260 Hz, where the fundamental wavelength is 2.7 mm, have probed the nature of the turbulent surface waves by tracing single particles and the relative motion of pairs of particles diffusing on the surface [10]. Especially has the relative diffusion of pairs been linked to the WWT theory [10,11]; however, the connection is made on a rather shaky basis [12]. The connection rests heavily on the exact form of the power spectrum; therefore we have undertaken measurements at a similar frequency. The result is shown in Fig. 3.

As seen, the slope at frequencies higher than the fundamental wave frequency of 110 Hz is close to the value predicted by WWT theory. At frequencies below the fundamental the spectrum is more or less constant. This result casts serious doubts on the validity of the derivation of the theory for pairs particle diffusion given in [11]. A corrected derivation that recovers the essential scaling relation is presented in [12].

Deviations From Theory

We also have performed measurements at even lower frequencies, down to 14 Hz. On the whole the main results are recovered. However, as seen in Fig. 4, we often encounter humps in the spectrum. These structures may be related to finite size effects. However, the fundamental wave frequency is here getting into the mixed gravity/capillary wave regime below the minimum wave velocity that occurs for waves of frequency around 14 Hz. Besides the energy cascade that results in the WWT spectrum, resonant energy transport directly may occur from a gravity wave to capillary waves of the same wave velocity [13]. Also, a wave number cascade to lower k values confuses the picture.

CONCLUSIONS

In conclusion, using laser-induced fluorescence, we have provided substantial and direct experimental support to the weak turbulence theory of capillary waves. Our measurements reproduce the WWT theory result for the spectrum over more than a decade within a few percent. Our measurements also involve a cross-over phenomenon in which the decay exponent for the surface-height fre-

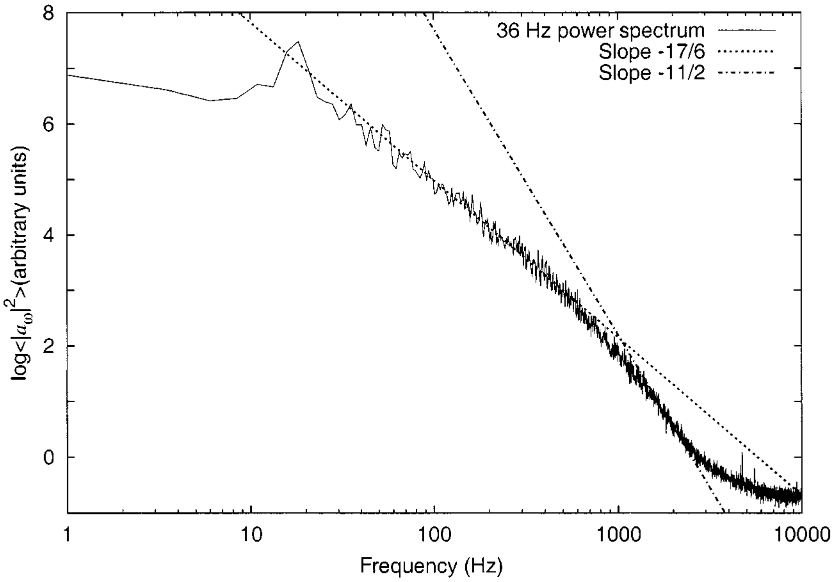

Fig. 2. Spectrum obtained at a drive frequency of 36 Hz. The straight lines have slopes −17/6 and −11/2, as predicted by theory.

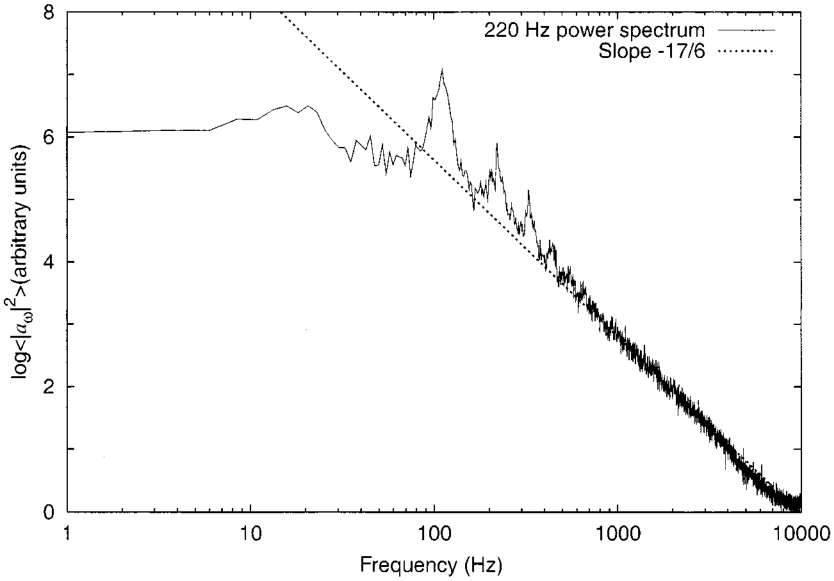

Fig. 3. Spectrum obtained at 220 Hz. The straight line has the slope −17/6, predicted by theory. Notice a relatively constant part below the fundamental wave frequency at 110 Hz.

quency spectrum changes from −17/6 to −11/2. If the extra wave number dependence introduced in the high-frequency regime by the wavelength becoming smaller than the diameter of the beam area is removed, the slope −17/6 is again recovered over more than a decade to within a few percent.

The special dissipation structures observed by Wright *et al.* [3,4] appear not to influence the scaling behavior of the frequency spectrum, although some of the structure seen in the spectra obtained may be related to intermittency. However, when averaged over many traces, these structures seem to disappear.

The interpretation of earlier experimental results on relative diffusion of pairs of particles at high (260 Hz) frequencies in terms of the WWT theory rested on the assumption of the form of the frequency spectrum being that given by the WWT theory. We have performed measurements of the spectrum also at these frequencies. The

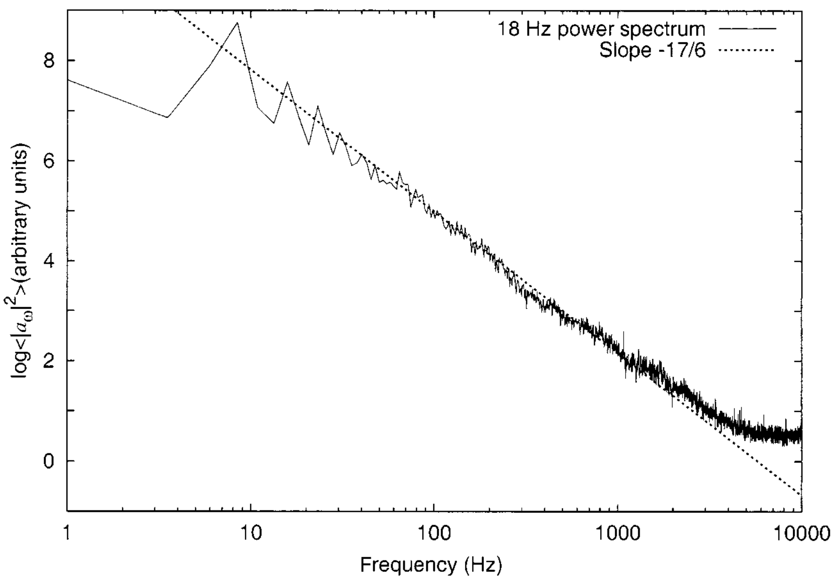

Fig. 4. Spectrum obtained at a drive frequency of 18 Hz.

results cast doubts on the validity of the derivation of the theoretical results used in the interpretation. Our measurements in the high-frequency regime are used in a rephrasing of the derivation [12] that recovers the main scaling exponent as given by the earlier derivation.

An interesting question that remains to be answered is to what extent the shape of the container (square or circular) influences the outcome of these kinds of experiments. The experiment was therefore repeated in a 15- by 19-cm square container, where bulk vorticity may be created by the corners. However, no change within the experimental uncertainty was detected in the outcome of the experiment.

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